Math 245B Lecture 7 Notes

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January 23, 2019

1 The Arzelà-Ascoli Theorem

1.1 Compactness of subsets of $C(X)$

Last time, we proved Tychonoff's theorem, which says that a product of compact spaces is compact. Really, we want to think about $\prod_{\alpha \in A} X_\alpha$ as the set of functions $f : A \to \bigcup_\alpha X_\alpha$ sending $\alpha \mapsto x_\alpha$, where $x_\alpha \in X_\alpha$ for all α. in analysis, finding compactness of spaces and subspaces of functions is very useful and important.

For this lecture, we will assume X is a compact, Hausdorff space. We will let $C(X) = C(X, \mathbb{C})$ (although everything here is true for $\mathbb{R}$ instead of $\mathbb{C}$. We also denote

$$\rho_u(f, g) = \|f - g\|_u = \sup_{x \in X} |f(x) - g(x)|.$$

We know that $(C(X), \rho_u)$ is a complete metric space. This is a big space. We will identify its compact subspaces. Let $\mathcal{F} \subseteq C(X)$. When is it closed and totally bounded? The point of this theorem is to give conditions for totally boundedness.

Definition 1.1. A family $\mathcal{F}$ is **equicontinuous at** $x \in X$ if for any $\varepsilon > 0$, there exists a neighborhood $U \ni x$ such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U$ and for all $f \in \mathcal{F}$. $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at every point.

This is the same neighborhood U for all $f \in \mathcal{F}$.

Definition 1.2. A family $\mathcal{F} \subseteq C(X)$ is **pointwise bounded** if the set $\{f(x) : f \in \mathcal{F}\}$ is bounded for all $x \in X$.

1.2 Statement and proof of the theorem

Theorem 1.1 (Arzelà-Ascoli)**.** *A subspace $\mathcal{F}$ is totally bounded if and only if it is equicontinuous and pointwise bounded. Moreover, $\overline{\mathcal{F}}$ is compact in $(C(X), \rho_u)$.*

Example 1.1. If X has a compact metric ρ, then $\mathcal{F}$ is equicontinuous if for all $x \in$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all y such that $\rho(x, y) < \delta$ (for all $f \in \mathcal{F}$).

Example 1.2. Any finite subset of $C(X)$ is equicontinuous.

Example 1.3. Let $X = [-1, 1]$. Let $\mathcal{F}$ be the sequence of functions which are 0 on $[-1, 0]$ and increase continuously to 1 (with steeper and steeper slope). This sequence converges to $\mathbb{1}_{(0,1]}$, which is not in $\mathcal{F}$. This family is not totally bounded, and the ever-increasing steepness at 0 makes this family not equicontinuous.

We will only prove one direction of the equivalence.[1]

Proof. ($\Longleftarrow$): Let $\varepsilon > 0$. We will cover $\mathcal{F}$ with finitely many subsets of ρ_u-diameter $< 4\varepsilon$. For every $x \in X$, there exists a neighborhood $U \ni x$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U_x$ and $f \in \mathcal{F}$. By compactness, there exists a finite subcover $X = U_{x_1} \cup \cdots \cup U_{x_m}$. For each $i = 1, \dots, m$, the set $\{f(x_i) : f \in \mathcal{F}\}$ is bounded. So $\bigcup_{i=1}^m \{f(x_i) : f \in \mathcal{F}\}$ is bounded. That is, there exists a finite $B \subseteq \mathbb{C}$ such that for all i and $f \in \mathcal{F}$, there exists a $z \in B$ such that $|f(x_i) - z| < \varepsilon$.

For each $\varphi \in B^m$, define $\mathcal{F}_\varphi = \{f \in \mathcal{F} : |f(x_i) - \varphi(x_i)| < \varepsilon \, \forall i = 1, \dots, m\}$. To finish, condier $f, g \in \mathcal{F}_\varphi$. We know that $|f(x_i) - g(x_i)| < 2\varepsilon$ fpr a;; $i = 1, \dots, n$. For any other $y \in X$, we have $y \in U_{x_i}$ for some i, so

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(y)| \\ &< \varepsilon + 2\varepsilon + \varepsilon \\ &= 4\varepsilon. \end{aligned}$$

So $\|f - g\|_u \leq 4\varepsilon$ for all $f, g \in \mathcal{F}_\varphi$. This gives total boundedness. $\square$

Remark 1.1. $\overline{\mathcal{F}}$ is compact because it is still totally bounded. In general, closure preseves total boundedness.

1.3 Alternate proof of Arzelà-Ascoli

Here is an alternative proof.

Proof. ($\Longleftarrow$): Let $\mathcal{F}$ be equicontinuous and pointwise bounded. Let D_x be a closed, bounded disc in the complex plane containing $\{f(x) : f \in \mathcal{F}\}$.We can think of $\mathcal{F} \subseteq \prod_{x \in X} D_x$, which is a compact product by Tychonoff's theorem. The following lemma (left as an exercise) completes the proof. $\square$

Lemma 1.1. *Let $\mathcal{F}$ be closed and equicontinuous in $C(X)$. Then*

1. *Restricted to $\mathcal{F}$, the uniform topology on $C(X)$ and the product topology on $\prod_{x \in X} D_x$ are the same.*

2. *$\mathcal{F}$ is also closed as a subset of $\prod_{x \in X} D_x$.*

[1] Professor Austin says that the other direction is not very useful, in his experience.

1.4 Arzelà-Ascoli in $\mathbb{R}^n$

In general, we can extend Arzelà-Ascoli to spaces that are not compact but made of countably many compact pieces. Let's see how this works in $\mathbb{R}^n$. Let $(f_n)_{n\in\mathbb{N}} \subseteq C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n)$. These are not even necessarily bounded. What is the appropriate notion of convergence for them?

Definition 1.3. The sequence $f_n \to f$ **locally uniformly** if $f_n|_K \to f|_K$ uniformly for all bounded $K \subseteq \mathbb{R}^n$.

Theorem 1.2. *If $(f_n)_{n\in\mathbb{N}} \subseteq C(\mathbb{R}^n)$ is equicontinuous and pointwise bounded, then there exist $f \in C(\mathbb{R}^n)$ and a subsequence $f_{n(k)} \to f$ locally uniformly as $k \to \infty$.*

Proof. For any $r \in \mathbb{N}$, Arzelà-Ascoli gives that $\{f_n : \overline{B_r(0)}\}$ is totally bounded in $C(\overline{B_r(0)})$. By a diagonal argument, there exists a subsequence $(f_{n(k)})_{k=1}^\infty$ such that $f_{n(k)}|_{\overline{B_r(0)}}$ converges uniformly to some $f^{(r)} \in C(\overline{B_r(0)})$. By the uniqueness of limits, we have $f^{(r)}_{\overline{B_r(0)}} = f^{(r)}$ for all $r < s$. So there exists a function $f : \mathbb{R}^n \to \mathbb{C}$ such that $f^{(r)} = f|_{\overline{B_r(0)}}$ for all r and $f \in C(\mathbb{R}^n)$. This is the same thing as locally uniform convergence $f_{n(k)} \to f$. $\square$

Remark 1.2. We can think of this proof as Arzelà-Ascoli applied to the image of $\mathcal{F}$ in $\prod_{r=1}^\infty C(\overline{B_r(0)})$.

Example 1.4. In $BC(\mathbb{R})$, consider $\{f_\alpha(x) = e^{i\alpha} : 1 \leq \alpha \leq 2\}$. This is pointwise bounded, and it is uniformly equicontinuous by the mean value theorem. Let $1 \leq \alpha < \beta \leq 2$. Then $\rho_u(f_\alpha, f_\beta) = 2$.